Scientific report- COST TD1207
STSM-TD1207-031216-081719

Abstract

In this document we provide a scientific report on the work of the STSM (Short Term Scientific Mission) TD1207-031216-081719 titled "K ring topology for energy distributions networks: the general case" carried out by Dr. Yoram Haddad at the Carnegie Mellon University, Pittsburgh,USA, hosted by Prof. Jon. We provide first some administrative details followed by the general purpose of the STSM in section 2. Then, in section 3 and 4 we present a detailed description of the work carried out. Future collaboration are presented in section 5. Finally confirmation of the mission achievement by the host can be found in the attached letter.

1 Administrative details

COST Action name: TD1207
STSM TITLE : K ring topology for energy distributions networks: the general case (STSM-TD1207-031216-081719)
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Dates:
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End date: Dec. 9, 2016

2 Purpose of the STSM

The first goal of this STSM was to complete a work initiated last year in a previous STSM which deal with energy distribution network. Last year we worked on some special case and this year we wanted to complete the work with the general case. In addition we wanted to visit the Center for Wireless and Broadband Networking, discuss other interesting research problem with Prof Peha, present some last results in a seminar and therefore made some last dissemination of TD1207 cost action which is close to completion.

3 General description on the work carried out

The general topic is linked to energy distribution. For this purpose we compared this problem with the somewhat equivalent one in the field of wireless networks. In the last year STSM we found that it is possible to connect an even number n of sites with $n/2$ rings, each of one including 2 sites and the aggregation node with a running time of $O(n^3)$. Although this is an important results, this might not be enough in general cases where there can be constraint with rings bigger than 3 nodes. Therefore we investigated in this STSM the general case where we have n/k rings, each one including the aggregation node and k cellular sites; $k3$. We did it in two steps. First we investigated the NP completeness of the problem. Secondly, we found a method to solve the problem. Background on the general topic and the particular case can be found in the last year's stsm report and can be provided upon request.

General case

$\frac{n}{k}$ rings, each one including the aggregation node and k cellular sites; $k \geq 3$
In this section, we assume the following:

- n is a multiple of k;
- the network includes $\frac{n}{k}$ rings, each one including the aggregation node and k cellular sites;
- for $1 \leq i \leq n$, p_i is the failure probability of the link OM_i;
- for $1 \leq i, j \leq n$, p_{ij} is the failure probability of the link M_iM_j;
- failure events are uncorrelated.

At first, we investigate the relation between the general maximum k-ring division problem and an NP-Complete problem.

Let $\mathcal{P}_k(n)$ be the set of k-combinations of $\{1, 2, \ldots, n\}$. Given a family of sets $F \subseteq \mathcal{P}_k(n)$ for $k \geq 3$, a k-set packing of $\{1, 2, \ldots, n\}$ is a set $S \subseteq F$ such that $\forall s_1, s_2 \in S, s_1 \cap s_2 = \varnothing$.

The maximum k-set packing problem (MSP) is to find a k-set packing S of $\{1, 2, \ldots, n\}$ such that for each k-set packing S' of $\{1, 2, \ldots, n\}$, $|S| \geq |S'|$. The corresponding decision problem ($d - MSP$) is a well-known NP-Complete problem [2], [1].

We define the maximum production $[0, 1)$ weighted k-set packing (MPWSP) as followed: given a family $F = \mathcal{P}_k(n)$ where $n = mk$ for some $m \in \mathbb{N}$ and a weight function $w : F \to [0, 1)$, the MPWSP problem is to find a k-set packing S of $\{1, 2, \dots, n\}$ such that for each k-set packing S' of F, $\prod_{u \in S} w(u) \geq \prod_{u \in S'} w(u)$.

Let $d - MPWSP$ denote the corresponding decision problem to MPWSP. $d - MSP$ is a particular case of $d - MPWSP$ with:

- $w(X) = 1$ for $X \in F$
- $w(X) = 0$ for $X \notin F$

Therefore, $d - MPWSP$ is as least as hard as $d - MSP$. Thus, d-MPWSP is NP-Hard (and in fact, d-MPWSP is NP-Complete).

Given an algorithm to solve the MPWSP problem, it can be used to solve the general maximum k-ring division problem as followed:
let $A^{max}_{i_1 i_2 \dots i_k}$ be the highest availability of all the rings including the k nodes i_1, i_2,..., i_k and the aggregation node:

$$A^{max}_{i_1 i_2 \dots i_k} = max(A_{j_1 j_2 \dots j_k} | j_1 j_2 \dots j_k \text{ is a permutation of } i_1 i_2 \dots i_k) \quad (1)$$

and

$$c_{i_1 i_2 \dots i_k} = log(A^{max}_{i_1 i_2 \dots i_k}) \quad (2)$$

An instance of MPWSP could be constructed by defining a family of sets $F = \mathcal{P}_k(n)$ and a weight function $w(i_1, i_2, , i_k) = A^{max}_{i_1 i_2 \dots i_k}$. Clearly, a solution to the constructed MPWSP instance yields a solution to the original maximum k-ring division problem.

Reciprocally, let us consider the following particular case:

$$V = U_1 \cup \dots \cup U_k$$
$$|U_1| = \dots = |U_k| = \frac{n}{k}$$
$$\forall u \in U_1, p_u = 0$$
$$\forall u \in U_2 \cup \dots \cup U_k, p_u = 1$$
$$\forall u_i \in U_i, \forall u_j \in U_j, |j - i| \neq 1 \to p_{ij} = 1$$
$$\forall u_i \in U_i, \forall u_j \in U_j, |j - i| = 1 \to p_{ij} \in [0, 1]$$

In this particular case, the aggregation node is connected to all the nodes of U_1 and no other node. Every connection between the aggregation node and anyone of the nodes of U_1

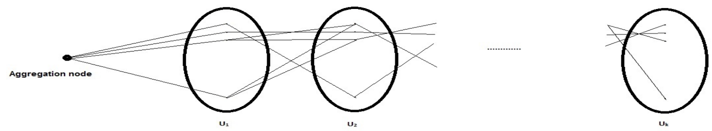

Figure 1: Particular case of k-ring division.

is assumed to be free of failure-risk. A node in a given subset U_i can be connected only to the nodes belonging to the adjacent sets U_{i-1} and U_{i+1}.

Then, the maximization of $c_{i_1 i_2 \ldots i_k}$ is a k-dimensional matching problem, which is known to be NP hard [3] for $k \geq 3$. Therefore, the general problem is NP hard for $k \geq 3$.

4 Approximation method

Since the general maximum k-ring division problem is NP-hard for $k \geq 3$, we propose hereafter an approximation method in order to converge to the solution.

Formalization as an Integer Linear Programming Problem

We can present the k-ring division problem as an ILP: the idea is to define binary variables which correspond to a k-ring.

$$P = \max \sum_{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n)} c_{i_1 i_2 \ldots i_k} x_{i_1 i_2 \ldots i_k} \tag{3}$$

subject to

$$\sum_{\substack{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \\ j \in \{i_1, i_2, \ldots, i_k\}}} x_{i_1 i_2 \ldots i_k} = 1, \forall j \in \{1, 2, \ldots, n\} \tag{4}$$

$$x_{i_1 i_2 \ldots i_k} \in \{0, 1\}, \forall \{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \tag{5}$$

The purpose of this method is to characterize the network topology by binary values: $x_{i_1 i_2 \ldots i_k} = 1$ if the nodes $i_1, i_2, \ldots, i_k$, together with the aggregation node, form a ring, and $x_{i_1 i_2 \ldots i_k} = 0$ else.

Constraints (4) and (5) forces each node j to be in exactly one k-ring.

General ILP is known to be NP-Hard [4]. However, linear programming can be solved in polynomial time. By replacing constraint (5) in (3) with the constraint:

$$x_{i_1 i_2 \ldots i_k} \geq 0, \forall \{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \tag{6}$$

(also known as LP relaxation) we get a polynomial-time solvable linear program.

Without loss of generality, we can assume that $c_{i_1 i_2 \dots i_k} > 0$ for each $\{i_1, i_2, \dots, i_k\} \in \mathcal{P}_k(n)$, since we can always add any constant to all the coefficients $c_{i_1 i_2 \dots i_k}$. Doing that does not change the set of vectors that maximizes the problem, because due to the constraints, each feasible vector contains exactly $\frac{n}{k}$ ones and $\binom{n}{k} - \frac{n}{k}$ zeros. Therefore, adding K to all the coefficients $c_{i_1 i_2 \dots i_k}$ is equivalent to adding the constant $K\frac{n}{k}$ to the original objective function.

Lemma 4.1. *Assuming $c_{i_1 i_2 \dots i_k} > 0$ for each $\{i_1, i_2, \dots, i_k\} \in \mathcal{P}_k(n)$, a vector x which maximizes the system*

$$P' = \max \sum_{\{i_1, i_2, \dots, i_k\} \in \mathcal{P}_k(n)} c_{i_1 i_2 \dots i_k} x_{i_1 i_2 \dots i_k} \tag{7}$$

subject to

$$\sum_{\substack{\{i_1, i_2, \dots, i_k\} \in \mathcal{P}_k(n) \\ j \in \{i_1, i_2, \dots, i_k\}}} x_{i_1 i_2 \dots i_k} \leq 1, \forall j \in \{1, 2, \dots, n\} \tag{8}$$

$$x_{i_1 i_2 \dots i_k} \in \{0, 1\}, \forall \{i_1, i_2, \dots, i_k\} \in \mathcal{P}_k(n) \tag{9}$$

is feasible to (3).

Proof. All we need to show is that $\sum_{\substack{\{i_1, i_2, \dots, i_k\} \in \mathcal{P}_k(n) \\ j \in \{i_1, i_2, \dots, i_k\}}} x_{i_1 i_2 \dots i_k} = 1, \forall j \in \{1, 2, \dots, n\}$. Assume by contradiction that there is a j for which $\sum_{\substack{\{i_1, i_2, \dots, i_k\} \in \mathcal{P}_k(n) \\ j \in \{i_1, i_2, \dots, i_k\}}} x_{i_1 i_2 \dots i_k} = 0$ (there is no other possibility since x satisfies constraint (9); this means that there is a node j that is not in any k-ring). Since n is a multiplier of k, there are $k - 1$ other nodes that are not in any k-ring, therefore, a new ring can be added to the sum contradicting the fact that x maximizes P'. □

Lemma 4.2. *Assuming $c_{i_1 i_2 \dots i_k} > 0$ for each $\{i_1, i_2, \dots, i_k\} \in \mathcal{P}_k(n)$, a vector x which maximizes (7) maximizes (3).*

Proof. Straight from Lemma (4.1) and from the fact that any feasible vector in (3) is a feasible vector in (7). □

However, not each solution to the relaxation yields a solution to the original problem. Consider the following example:

$$n = 6; k = 3 \tag{10}$$

$$c_{124} = c_{135} = c_{236} = c_{456} = 1; \text{ all other } c_{ijl} = 0 \tag{11}$$

Then, $\max \sum c_{ijl}x_{ijl}$ subject to (4), (6) is obtained only for:

$$x_{124} = x_{135} = x_{236} = x_{456} = \frac{1}{2};\ \text{all other } x_{ijl} = 0 \tag{12}$$

Conclusion

Availability is maximized when the number of rings is high and the ring size distribution is regular. In this paper, we show that the partition of a network including an aggregation node and n cellular sites into $\frac{n}{2}$ rings, each one including the aggregation node and 2 cellular sites, can be solved in a time of $O(n^3)$. Regarding a partition with larger rings, the problem is similar to a k-set partition problem, which is NP-hard for $k \geq 3$. We propose an approximation method, based on linear programming to accelerate the convergence.

5 Future collaboration

As mentioned by prof Peha in the confirmation host letter, the very fruitful discussion we had, motivate us to meet again in the future, maybe hosting Prof. Peha in our institute. In addition we seriously consider the possibility to apply for a bilateral grant in the next period.

6 Host section

See attached letter

References

[1] Elad Hazan, Shmuel Safra, and Oded Schwartz. On the complexity of approximating k-set packing. *computational complexity 15.1*, pages 20–39, 2006.

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[4] Alexander Schrijver. *Theory of linear and integer programming.* Wiley, 1998.